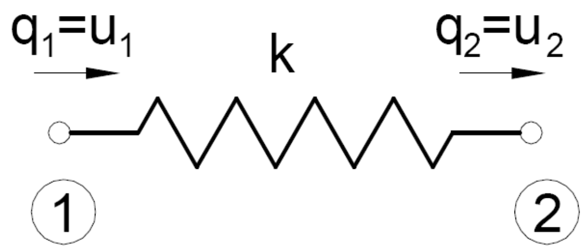
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 2C. Spring type element

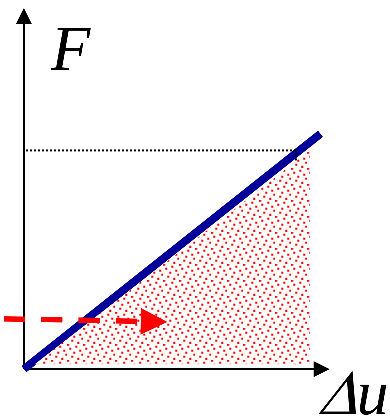
03.2025

Local spring stiffness matrix



vector of nodal parameters:

$$\underset{2\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



$$F = k \cdot \Delta u = k \cdot (u_2 - u_1)$$

elastic energy of the element:

$$\boxed{U_e = \frac{1}{2} F \Delta u} = \frac{1}{2} k (\Delta u)^2 = \frac{1}{2} k (u_2 - u_1)(u_2 - u_1).$$

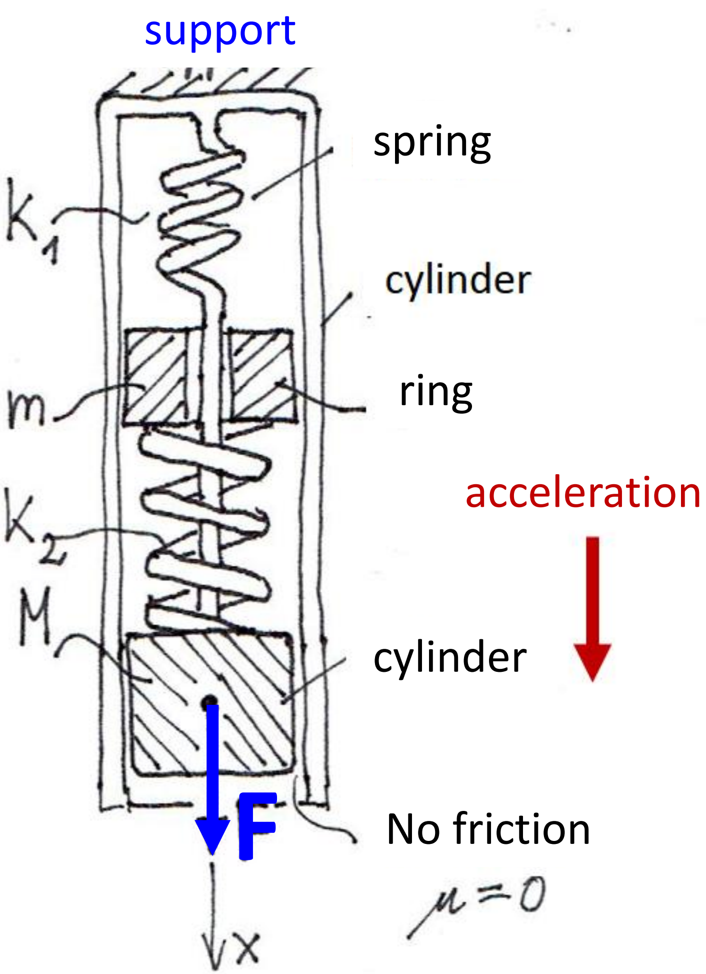
$$U_e = \frac{1}{2} \lfloor u_1, u_2 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boxed{U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e}$$

element stiffness matrix:

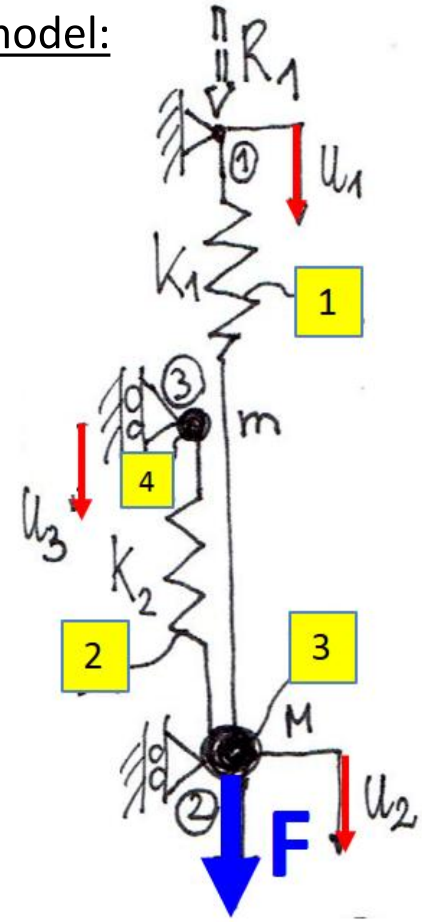
$$\boxed{[k]_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

Example: Build a FEM model. Find the elastic strain energy and potential load energy. Calculate the displacements and reaction



(*Weightless springs*)

FE model:



- finite elements

- nodes

$NOE = 4$

$NON = 3$

$n_p = 1$

$NDOF = 3 \cdot 1 = 3$

$$\underset{\substack{NDOF \times 1 \\ \shortparallel \\ 3}}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\underset{3\times 1}{\{F\}^n} = \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix}$$

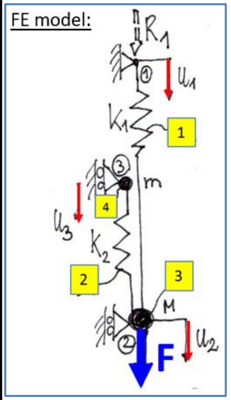
No load at node 3

$$U = \frac{1}{2} \underset{1\times 3}{\lfloor q \rfloor} \underset{3\times 3}{[K]} \underset{3\times 1}{\{q\}}$$

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \underset{3\times 1}{\{F\}}$$

Local notation

Element 1



$1' = 1$

$2' = 2$

$$\underset{1\times2}{\lfloor q \rfloor_1} = \lfloor q_1, q_2 \rfloor_1 = \lfloor u_1, u_2 \rfloor_1$$

$$(ne) \quad \underset{1\times3}{\lfloor q \rfloor} = \lfloor \underbrace{u_1, u_2}_{\lfloor q \rfloor_1}, u_3 \rfloor$$

$$\lfloor F^x \rfloor_1 = \lfloor F_{11}, F_{21} \rfloor_1 = \lfloor 0, 0 \rfloor_1$$

$$\underset{1\times3}{\lfloor F^x \rfloor_1^*} = \lfloor F_{11}, F_{21}, 0 \rfloor = \lfloor 0, 0, 0 \rfloor$$

$$\underset{2\times2}{[k]_1} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$[k]_1^* = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 2

$$\underset{1\times2}{\lfloor q \rfloor_2} = \lfloor q_1, q_2 \rfloor_2 = \lfloor u_3, u_2 \rfloor_2$$

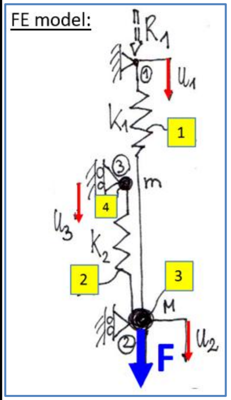
$$\underset{1\times3}{\lfloor q \rfloor} = \lfloor u_1, u_2, u_3 \rfloor$$

$$\underset{1\times2}{\lfloor F^x \rfloor_2} = \lfloor F_{12}, F_{22} \rfloor_2 = \lfloor 0, 0 \rfloor_2$$

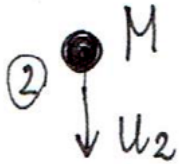
$$\lfloor F^x \rfloor_2^* = \lfloor 0, F_{22}, F_{12} \rfloor = \lfloor 0, 0, 0 \rfloor$$

$$\underset{2\times2}{[k]_2} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$[k]_2^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$



Element 3



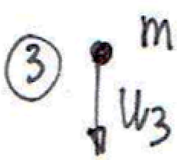
$$\lfloor q \rfloor_{1\times 3} = \lfloor u_1, \boxed{u_2}, u_3 \rfloor$$

$$[k]^*_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_3 = M \cdot a_x$$

$$\lfloor F^x \rfloor^*_{3\;(1\times 3)} = \lfloor 0, F^x_3, 0 \rfloor = \lfloor 0, Ma_x, 0 \rfloor$$

Element 4

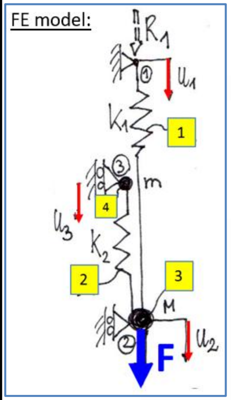


$$\lfloor q \rfloor_{1\times 3} = \lfloor u_1, u_2, \boxed{u_3} \rfloor$$

$$[k]^*_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_4 = m \cdot a_x$$

$$\lfloor F^x \rfloor^*_{4\;(1\times 3)} = \lfloor 0, 0, F^x_4 \rfloor = \lfloor 0, 0, ma_x \rfloor$$



Global stiffness matrix :

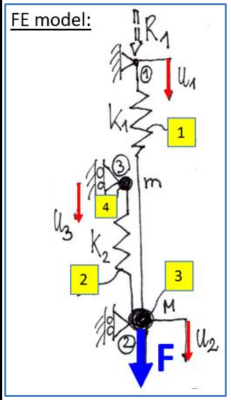
$$[K]_{3\times3} = \sum_{e=1}^{4} [k]_e^* = \left[\begin{array}{c|c|c} k_1+0+0+0 & -k_1+0+0+0 & 0+0+0+0 \\ \hline -k_1+0+0+0 & k_1+0+0+k_2 & 0+0+0-k_2 \\ \hline 0+0+0+0 & 0+0+0-k_2 & 0+0+0+k_2 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} k_1 & -k_1 & 0 \\ \hline -k_1 & k_1+k_2 & -k_2 \\ \hline 0 & -k_2 & k_2 \end{array}\right]$$

Elastic strain Energy:

$$U = \frac{1}{2} \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

(we need u_2 and u_3 $(u_1=0)$ to find the value of U.)



$$\lfloor F \rfloor_e = \lfloor F^X \rfloor_e + \cancel{\lfloor F^p \rfloor_e} = \lfloor F^X \rfloor_e \Rightarrow \lfloor F \rfloor_e^* = \lfloor F^X \rfloor_e^*$$

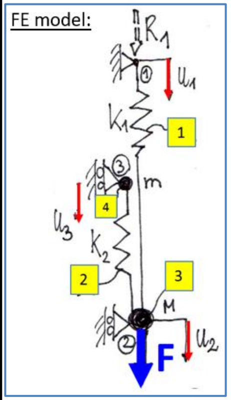
(no surface load)

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \underset{3\times 1}{\{F\}}$$

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\sum_{e=1}^{4} \{F\}_e^* + \{F\}^n \right) = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\begin{Bmatrix} 0+0+0+0 \\ 0+0+Ma_x+0 \\ 0+ma_x+0+0 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix} \right) =$$

$$= \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{Bmatrix} R_1 \\ Ma_x + F \\ ma_x \end{Bmatrix} \underset{(u_1=0)}{=\!=} u_2 \cdot (Ma_x + F) + u_3 \cdot ma_x$$

(we need u_2 and u_3 to find the value of W.)



Solution:

$$\boxed{V = \frac{1}{2}\lfloor q \rfloor [K]\cdot\{q\} - \lfloor q \rfloor \cdot \{F\}} \longrightarrow \min \longrightarrow \underset{3\times 3}{[K]}\cdot\underset{3\times 1}{\{q\}} = \underset{3\times 1}{\{F\}}$$

$$\boxed{u_1 = 0} \longrightarrow \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ Ma_x + F \\ ma_x \end{Bmatrix}$$

$$\underset{2\times 2}{[K]}\cdot\underset{2\times 1}{\{q\}} = \underset{2\times 1}{\{F\}}$$

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} Ma_x + F \\ ma_x \end{Bmatrix}$$

$$\begin{cases} (k_1+k_2)\cdot u_2 - k_2\cdot u_3 = Ma_x + F \\ -k_2\cdot u_2 + k_2\cdot u_3 = ma_x \end{cases} \Longrightarrow u_2,\ u_3$$

Reaction:

$$\left(\text{1st row of } \underset{3\times 3}{[K]}\right)\cdot \underset{3\times 1}{\{q\}} = R_1 \Rightarrow R_1 = k_1\cdot 0 - k_1\cdot u_2 + 0\cdot u_3 = -k_1 u_2$$